ON A FORM OF STEADY WAVES OF FINITE AMPLITUDE

PMM Vol. 32, №6, 1968, pp. 1014-1025
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(Received June 18, 1968)

The problem of steady waves of finite amplitude produced by pressure distributed periodically over the surface of a stream of heavy fluid of finite constant depth is considered.

This problem for a stream of infinite depth was first posed and solved approximately by Sretenskii [1] in 1953. The problem is solved rigorously in [2] for an infinitely deep stream and for a more general pressure distribution over the surface. As in the case of an infinitely deep stream, the pressure is specified in the form of some infinite trigonometric series. We also investigate the special case when the wavelength of the given pressure coincides with the length of the steady free wave corresponding to the specified velocity and constant pressure at the surface. The results of the present paper appear in condensed form in [3].

1. Formulation of the problem and derivation of the basic integral equation. Let us consider the plane-parallel steady motion of an ideal incompressible heavy fluid of finite constant depth h bounded by a free surface on top; the pressure $p = p_0(x)$ at this surface is a given periodic function of the horizontal coordinate x. We assume that at this horizontal bottom surface the stream moves with a given constant average velocity c directed from left to right.

Owing to the periodically distributed pressure, the surface of the stream assumes the shape of a stationary periodic wave in the coordinates attached to the progressing wave moving with the velocity c.

Let the required wave and the pressure $p_0(x)$ be equally symmetric with respect to the crest vertical. We direct the y-axis along the axis of symmetry (vertically upward) and place the origin O at the point of intersection of the y-axis with the free surface, and direct the x-axis towards the right.

We choose the flow plane xy as the plane of the complex variable $z = x + iy$.

We introduce the usual notation: φ is the velocity potential, ψ is the stream function, $w = \varphi + i\psi$ is the complex velocity potential, and U, V are the projections of the velocity vector q on the coordinate axes. This means that

$$\frac{dw}{dz} = -U + iV, \qquad U = -\frac{\partial\varphi}{\partial x}, \quad V = -\frac{\partial\varphi}{\partial y}$$

In order to derive the basic equation of the problem from the boundary condition we conformally map the domain occupied by a single wave (which takes the form of a vertical rectangle bounded on the top by a wavy curve) onto the rectangle

$$|\varphi| \leqslant {}^1/_2 c\lambda, \qquad 0 < \psi < \psi_0$$

(here $\psi = \psi_0$ is the stream discharge per unit time). We then conformally map this rectangle onto the interior of a circular ring with its center at the origin of the plane $u = u_1 + iu_2$. We assume here that the wavelength λ coincides with the period of the function $p_0(x)$.

As we know, the latter mapping is given by Formula

$$w = \frac{\lambda c}{2\pi i} \ln u \qquad (1.1)$$

Here the segment $|\varphi| \leqslant {}^1/_2\, c\lambda$ corresponding to a free surface becomes the outer circle of unit radius, while the segment corresponding to the bottom becomes the inner circle of radius

$$r_0 = \exp(-2\pi\psi_0/\varphi_0) = \exp(-2\pi h/\lambda)$$

smaller than unity. The ring is slotted along the segment $(-1, -r_0)$.

The mapping of this ring in the plane u onto the domain of a single wave in the plane z can be found from the relation

$$\frac{dz}{du} = -\frac{\lambda}{2\pi i}\frac{f(u)}{u} \tag{1.2}$$

The function $f(u)$ can be expressed as a Laurent series inside the ring in the plane u under consideration. The coefficients of this series must be real by virtue of the symmetry of the wave; the boundary condition of nonleakage at the bottom is fulfilled.

Making use of the Bernoulli integral at the surface, we convert to the variable u in the latter and set $u = e^{i\theta}$; recalling that $p = p_0(x)$ at the surface, on differentiating with respect to θ we have

$$\frac{1}{\rho}\frac{dp_0}{dx}\frac{dx}{d\theta} = -g\frac{dy}{d\theta} - \frac{1}{2}\frac{dq^2}{d\theta} \tag{1.3}$$

Here θ is the angle between the radius vector in the plane u and the axis u_1; ρ is the density; g is the gravitational acceleration; q is the absolute value of the velocity vector.

As usual, by introducing the function [4]

$$\omega(u) = \Phi + i\tau = -i \ln f(u) \tag{1.4}$$

we find from (1.4) and (1.2) that for $u = e^{i\theta}$

$$\frac{dx}{d\theta} + i\frac{dy}{d\theta} = -\frac{\lambda}{2\pi} e^{-\tau(\theta)}(\cos\Phi + i\sin\Phi) \tag{1.5}$$

Formulas (1.4), (1.2) and (1.1) imply that Φ is equal to the angle between the velocity vector $\mathbf{q}$ and the x-axis everywhere in the stream, and that

$$q = ce^{\tau} \tag{1.6}$$

By virtue of (1.5), (1.6), Eq. (1.3) yields a differential relation which we integrate, replacing the integration constant by the parameter

$$\mu = \frac{3g\lambda}{2\pi c^3} e^{-3\tau(0)} \tag{1.7}$$

which is related to the additive constant in $p_0(x)$.

Taking the logarithmic derivative of both sides of the resulting integral relation, we obtain

$$\frac{d\tau}{d\theta} = \frac{\mu[\sin\Phi + Q(\theta)\cos\Phi]}{3}\left[1 + \mu\int_0^\theta(\sin\Phi + Q\cos\Phi)\,d\eta\right]^{-1} \tag{1.8}$$

where

$$Q(\theta) = \frac{1}{\rho g}\frac{dp_0}{dx} \tag{1.9}$$

Equation (1.8) gives us the relationship on the circle $|u| = 1$ between the real and imaginary parts of the analytic function $\omega(u)$ which is regular inside the ring whose outer boundary is the circle $|u| = 1$.

From the theory of analytic functions we know that $\Phi(\theta)$ and $d\tau/d\theta$ are related by a Dini relation of the form

$$\Phi(\theta) = 3\int_0^{2\pi}\frac{d\tau}{d\eta}K(\eta,\theta)\,d\eta, \qquad K(\eta,\theta) = \sum_{n=1}^{\infty}\frac{\varphi_n(\eta)\varphi_n(\theta)}{\nu_n} \tag{1.10}$$

where the eigenfunctions $\varphi_n(\theta)$ and the eigenvalues ν_n of the kernel $K(\eta,\theta)$ are given by Formulas

$$\varphi_n(\theta) = \frac{\sin n\theta}{\sqrt{\pi}}, \qquad \nu_n = 3n\,\mathrm{cth}\left(2\pi n\frac{h}{\lambda}\right) \tag{1.11}$$

From (1.8) and (1.10) we finally obtain

$$\Phi(\theta) = \mu \int_0^{2\pi} H[\Phi(\eta), \eta]\left[1 + \mu \int_0^{\eta} H[\Phi, \eta_1]\, d\eta_1\right]^{-1} K(\eta, \theta)\, d\eta$$
$$H[\Phi(\eta), \eta] = \sin \Phi(\eta) + Q(\eta) \cos \Phi(\eta) \qquad (1.12)$$

This is the integral equation of the problem. For p_0 = const this equation yields Nekrasov's equation [4] for a finite depth.

In solving Eq. (1.12) we assume that

$$Q(\theta) = \frac{1}{\rho g}\frac{dp_0}{dx} = \sum_{n=1}^{\infty} \varepsilon^n d_n \sin n\theta \qquad (1.13)$$

where ε is a small dimensionless positive parameter; d_n are given real numbers; the infinite series

$$\varepsilon |d_1| + \varepsilon^2 |d_2| + \varepsilon^3 |d_3| + \ldots$$

converges in a disk of radius $\varepsilon_0 > 0$.

We note that p_0 in the initial problem is a periodic function of x specified to within an additive constant. It can be shown that solving our problem under condition (1.13) is equivalent to specifying the series

$$\frac{1}{\rho g}\frac{dp_0}{dx} = -\sum_{n=1}^{\infty} \varepsilon^n c_n' \sin \frac{2\pi n}{\lambda} x, \qquad c_n' = \sum_{m=0}^{\infty} \varepsilon^m c_{mn}'$$

Here we can either assume that the coefficients c_{0n}' are given and use them to determine the d_n, or, conversely, we can determine the coefficients c_{mn}' $(m = 1, 2, \ldots)$ in terms of the d_n. If we set $d_n = d_{0n} + \varepsilon d_{1n} + \varepsilon^2 d_{2n} + \cdots$ (this will not be our approach), then either the c_{mn}' $(m = 1, 2, \ldots)$ can be assumed given and used to determine d_{in} $(i = 1, 2, \ldots)$, or vice versa.

The parametric equation of the wave profile can be obtained from (1.5) in the form

$$x = -\frac{\lambda}{2\pi}\int_0^{\theta} e^{-\tau(\eta)} \cos \Phi(\eta)\, d\eta, \qquad y = -\frac{\lambda}{2\pi}\int_0^{2\pi} e^{-\tau(\eta)} \sin \Phi(\eta)\, d\eta \qquad (1.14)$$

Formulas (1.14) indicate that in solving the problem we must determine not only Φ but also $\tau(\theta)$. These functions are given by the following trigonometric series:

$$-\tau(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta, \qquad \Phi(\theta) = \sum_{n=1}^{\infty} B_n \sin n\theta \qquad (1.15)$$

Expansion of the function $\ln f(u) = i\omega(u)$ in a Laurent series yields the following relations between the coefficients of these series:

$$A_n = \frac{\nu_n}{3n} B_n \qquad (n = 1, 2, 3 \ldots) \qquad (1.16)$$

Thus, if we know B_n, we can find all of the A_n except A_0.

Let us transform Formula (1.7). Setting

$$\mu_0 = \frac{3}{2}\frac{g\lambda}{\pi c^2} \qquad (1.17)$$

we find from Eqs. (1.15), (1.17) and (1.7) that

$$\mu = \mu_0 \exp\left[3\left(A_0 + \sum_{n=1}^{\infty} A_n\right)\right] \qquad (1.18)$$

Setting $\theta = 2\pi$ in the right side of the first formula of (1.14), we obtain $-\lambda$, since the resulting decrease in x is equal to λ. In this way we arrive at the following equation for determining A_0:

$$\exp(-A_0) = \frac{1}{2\pi}\int_0^{2\pi} \exp[-\tau(\eta) - A_0] \cos\Phi(\eta)\, d\eta \tag{1.19}$$

where $-\tau(\eta) - A_0$ does not contain A_0 by virtue of (1.15).

Setting

$$\Psi(\theta) = \left[1 + \mu\int_0^\theta H[\Phi, \eta]\, d\eta\right]^{-1} \tag{1.20}$$

we reduce Eq. (1.12) (as in the case of infinite depth [2]) to an equivalent system of two equations with the unknown functions $\Phi(\theta)$ and $\Psi(\theta)$. To this end we differentiate (1.20) with respect to θ to obtain $\Psi'(\theta) = -\mu\Psi^2(\theta)\, H[\Phi(\theta), \theta]$

Integrating both sides of this equation over θ and noting that for $\theta = 0$ it is necessarily the case that $\Psi(0) = 1$, we obtain

$$\Psi(\theta) = 1 - \mu\int_0^\theta \Psi^2(\eta)\, H[\Phi, \eta]\, d\eta \tag{1.21}$$

By virtue of (1.20) Eq. (1.12) becomes

$$\Phi(\theta) = \mu\int_0^{2\pi} K(\eta, \theta)\, H[\Phi, \eta]\, \Psi(\eta)\, d\eta \tag{1.22}$$

It is easy to verify that system (1.21),(1.22) is equivalent to single equation (1.12). Let us reduce Eq. (1.18) to its final form. We set

$$\mu = \mu_0\left\{\exp\left[3\left(A_0 + \sum_{n=1}^{\infty} A_n\right)\right] - 1 + 1\right\} = \mu_0(1 + \mu') \tag{1.23}$$

$$\mu' = \exp\left[3\left(A_0 + \sum_{n=1}^{\infty} A_n\right)\right] - 1 \tag{1.24}$$

From (1.19) and (1.15) we obtain

$$\exp(-3A_0) = \left[\frac{1}{2\pi}\int_0^{2\pi} \exp\left(\sum_{n=1}^{\infty} A_n \cos n\eta\right) \cos\Phi(\eta)\, d\eta\right]^3 \tag{1.25}$$

By virtue of (1.25) Eq. (1.24) finally becomes

$$\mu' = \exp\left(3\sum_{n=1}^{\infty} A_n\right)\left[\frac{1}{2\pi}\int_0^{2\pi} \exp\left(\sum_{n=1}^{\infty} A_n \cos n\eta\right) \cos\Phi(\eta)\, d\eta\right]^{-3} - 1 \tag{1.26}$$

Thus, we have reduced the problem to the determination of the functions $\Phi(\theta, \varepsilon)$ and $\Psi(\theta, \varepsilon)$ from system (1.21),(1.22), the parameter $\mu(\varepsilon)$ from (1.26),(1.23), and the coefficient $A_0(\varepsilon)$ from (1.19). There are two cases to be considered: $\mu_0 \neq \nu_n$ and $\mu_0 = \nu_n$.

In the next two Sections we show that in the first case the solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, $\mu(\varepsilon)$, $A_0(\varepsilon)$ is constructed in the form of series in whole powers of the parameter ε. The second case is illustrated by considering the value $\mu_0 = \nu_1$.

The solution is obtained here in the form of series in powers of $\varepsilon^{1/2}$. In both cases we use the methods of Liapunov and Schmidt [5] to prove that these series converge absolutely and uniformly for $0 \leqslant \theta \leqslant 2\pi$ and for small values $|\varepsilon| < \varepsilon_1 < \varepsilon_0$ and that they yield the unique solution of the problem which is small with respect to ε and continuous in θ (Theorems 1 and 2; here ε_1 is the smaller of the numbers ε_3' and ε_4'').

2. The solution in the case $\mu_0 \neq \nu_n$. In the so-called regular case system of equations (1.22), (1.21) can be rewritten as

$$\Phi(\theta) = \mu_0(1+\mu') \int_0^{2\pi} K(\eta, \theta)[\Phi(\eta) + P(\eta)]\, d\eta \tag{2.1}$$

$$\Psi^*(\theta) = -\mu_0(1+\mu') \int_0^{\theta} \{\Phi(\eta) + \varepsilon d_1 \sin\eta + F_1[\Phi(\eta), \Psi^*(\eta), \varepsilon]\}\, d\eta$$

where

$$\Psi^*(\theta) = \Psi(\theta) - 1, \qquad P(\eta) = \varepsilon d_1 \sin\eta + F[\Phi(\eta), \Psi^*(\eta), \varepsilon] \tag{2.2}$$

$$F[\Phi(\eta), \Psi^*(\eta), \varepsilon] = \sin\Phi(\eta) - \Phi(\eta) + Q(\eta)(\cos\Phi - 1) +$$
$$+ Q(\eta) - \varepsilon d_1 \sin\eta + \Psi^*(\eta) H[\Phi(\eta), \eta], \quad F_1[\Phi(\eta), \Psi^*(\eta), \varepsilon] =$$
$$= Q\cos\Phi - \varepsilon d_1 \sin\eta + \sin\Phi - \Phi + [\Psi^{*2}(\eta) + 2\Psi^*(\eta)] H[\Phi(\eta), \eta]$$

Let us transform the first equation of (2.1). It is clearly equivalent to

$$\Phi(\theta) + P(\theta) = \mu_0(1+\mu') \int_0^{2\pi} K(\eta, \theta)[\Phi(\eta) + P(\eta)]\, d\eta + P(\theta) \tag{2.3}$$

We denote the resolvent of the linear integral equation with the kernel $K(\eta, \theta)$ and the parameter $\mu_0(1+\mu')$ by $R[\theta, \eta, \mu_0(1+\mu')]$. Then, following the Liapunov-Schmidt method, we can rewrite Eq. (2.3), and therefore (2.1), in the following equivalent form:

$$\Phi(\theta) = \mu_0(1+\mu') \int_0^{2\pi} R[\eta, \theta, \mu_0(1+\mu')] P(\eta)\, d\eta \tag{2.4}$$

Let us express the resolvent R in explicit form. To this end we make use of a formula for the resolvent $\Gamma(x, y; \lambda)$ which is valid in the case of a symmetric kernel $K(x, y)$ (e.g. see Goursat [6])

$$\Gamma(x, y; \lambda) = K(x, y) + \sum_{n=1}^{\infty} \lambda \frac{\varphi_n(x)\varphi_n(y)}{\lambda_n(\lambda_n - \lambda)}$$

In this case $x = \eta$, $y = \theta$, $\lambda = \mu_0(1+\mu')$, $\lambda_n = \nu_n$ the resolvent is denoted by R and the kernel $K(\eta, \theta)$ is given by Formula (1.10). We therefore have

$$R[\eta, \theta; \mu_0(1+\mu')] = \frac{1}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\sin n\eta \sin n\theta}{\nu_n - \mu_0(1+\mu')} \tag{2.5}$$

To reduce Eq. (2.4) to its final form, we substitute into it expression $P(\eta)$ from (2.2),

$$\Phi(\theta) = \varepsilon d_1 \mu_0(1+\mu') \int_0^{2\pi} R[\eta, \theta, \mu_0(1+\mu')] \sin\eta\, d\eta +$$

$$+ \mu_0(1+\mu') \int_0^{2\pi} R[\eta, \theta, \mu_0(1+\mu')] F[\Phi(\eta), \Psi^*(\eta), \varepsilon]\, d\eta$$

Making use of Formula (2.5), we obtain

$$\varepsilon d_1\mu_0(1+\mu')\int_0^{2\pi} R[\eta,\theta,\mu_0(1+\mu')]\sin\eta\,d\eta = \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}$$

The preceding equation therefore assumes the following final form:

$$\Phi(\theta) = \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}\sin\theta +$$

$$+\mu_0(1+\mu')\int_0^{2\pi} R[\eta,\theta,\mu_0(1+\mu')]\,F[\Phi(\eta),\Psi^*(\eta,\varepsilon)]\,d\eta \qquad (2.6)$$

Let us transform the second equation of system (2.1). Carrying out the integrations and self-evident transformations in its right side, we obtain

$$\Psi^*(\theta) = \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}(\cos\theta-1) -$$

$$-\mu_0(1+\mu')\int_0^{\theta}\left[\Phi(\eta) - \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}\sin\eta\right]d\eta -$$

$$-\mu_0(1+\mu')\int_0^{\theta} F_1[\Phi(\eta),\Psi^*(\eta),\varepsilon]\,d\eta \qquad (2.7)$$

We begin by solving system (2.6), (2.7), assuming that the parameters ε and μ' are real given quantities of small absolute value. We then define $\mu'(\varepsilon)$ in such a way as to ensure fulfillment of relation (1.26).

Solving system (2.6), (2.7) by the method of successive approximations, we take as our first approximation

$$\Phi^{(1)}(\theta) = \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}\sin\theta \qquad (2.8)$$

$$\Psi^{*(1)}(\theta) = \mu_0(1+\mu')\varepsilon d_1(\cos\theta-1) + \mu_0(1+\mu')\int_0^{\theta}\Phi^{(1)}(\eta)\,d\eta =$$

$$= \frac{\mu_0\nu_1(1+\mu')\varepsilon d_1}{\nu_1-\mu_0(1+\mu')}(\cos\theta-1) \qquad (2.9)$$

Any kth approximation ($k>1$ and finite) is given by Formulas

$$\Phi^{(k)}(\theta) = \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}\sin\theta +$$

$$+\mu_0(1+\mu')\int_0^{2\pi} R[\eta,\theta,\mu_0(1+\mu')]\,F[\Phi^{(k-1)},\Psi^{*(k-1)},\varepsilon]\,d\eta \qquad (2.10)$$

$$\Psi^{*(k)}(\theta) = \frac{\nu_1\mu_0(1+\mu')\varepsilon d_1}{\nu_1-\mu_0(1+\mu')}(\cos\theta-1) -$$

$$-\mu_0(1+\mu')\int_0^{\theta}\left[\Phi^{(k)}(\eta) - \frac{\varepsilon d_1\mu_0(1+\mu')}{\nu_1-\mu_0(1+\mu')}\sin\eta\right]d\eta -$$

$$-\mu_0(1+\mu')\int_0^{\theta} F_1[\Phi^{(k-1)}(\eta),\Psi^{*(k-1)}(\eta),\varepsilon]\,d\eta \qquad (2.11)$$

We can show by mathematical induction that the quantities $\Phi^{(l)}(\theta), \Psi^{*(l)}(\theta)$ are analytic functions in ε and μ' for $|\varepsilon| < \varepsilon_0$ and $|\mu'| < \mu^0$.

We can show in the usual way that the successive approximations converge to the solution $\Phi(\theta, \varepsilon, \mu'), \Psi^*(\theta, \varepsilon, \mu')$ of system (2.6), (2.7). This solution is the unique solution of the system which is small with respect to ε and μ' and continuous in θ Both functions are analytic with respect to ε and μ' for $|\varepsilon| < \varepsilon_1' \leqslant \varepsilon_0$ and $|\mu'| < \mu^0$ In order to construct Eq. (1.26) in explicit form we must verify the convergence of the series $A_1 + A_2 + A_3 + \ldots$ To do this we must find $\tau(\theta)$ from (1.15). The foregoing implies that the derivative $d\tau / d\theta$ defined by Formula (1.8) is a continuous function in θ for $0 \leqslant \theta \leqslant 2\pi$ and an analytic function in ε and μ'. The function $\tau(\theta)$ is therefore expressible as the absolutely and uniformly convergent first series of (1.15); its terms and sum are analytic functions of ε and μ' for $|\varepsilon| < \varepsilon_1'$ and $|\mu'| < \mu^0$. Moreover, the second series of (1.15) for $\Phi(\theta)$ has the same properties. Thus, the series $A_1 + A_2 + \ldots$ is convergent and the right side of Eq. (1.26) is an analytic function of ε and μ'. For $\varepsilon = 0$ Eq. (1.26) has the solution $\mu' = 0$. On the other hand, transposing all terms to the left side in Eq. (1.26) and denoting it by $G(\varepsilon, \mu')$, we obtain

$$G(\varepsilon, \mu') = 0 \qquad (2.12)$$

Since

$$\left(\frac{\partial G}{\partial \mu'}\right)_{\varepsilon=\mu'=0} = 1$$

it follows by the theorem on implicit functions that Eq. (2.12), and therefore (1.26), has the unique solution $\mu'(\varepsilon)$ for small values of ε. This solution satisfies the condition

$$\lim \mu'(\varepsilon) = 0 \qquad (\varepsilon \to 0)$$

and is an analytic function of ε for $|\varepsilon| < \varepsilon_2' \leqslant \varepsilon_1'$. Substituting in this expression for $\mu'(\varepsilon)$, we find that $-\tau(\theta) - A_0$ and $\Phi(\theta)$ are series in powers of ε. Hence, by (1.19) the expression $\exp(-A_0) = \chi(\varepsilon)$ is also an analytic function of ε, and, since by hypothesis $\lim A_0 = 0$ as $\varepsilon \to 0$, it follows that $\chi(0) = 1$. Hence, $-A_0 = \ln \chi(\varepsilon)$ is also an analytic function of ε for $|\varepsilon| < \varepsilon_3' \leqslant \varepsilon_2'$.

On the other hand, we note that the parameter μ' in Eqs. (2.6) and (2.7) had fixed small absolute values. Having found μ' from (1.23), we can find μ. We have therefore proved the following theorem.

Theorem 1. System (1.22), (1.21), (1.26), (1.19) for $\mu_0 \neq \nu_n$ has a uniqe solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, $\mu(\varepsilon)$ $A_0(\varepsilon)$ which is small with respect to ε and continuous in θ $(0 \leqslant \theta \leqslant 2\pi)$. This solution is an analytic function of ε for $|\varepsilon| < \varepsilon_3'$.

This theorem implies that in solving system (1.22), (1.21), (1.26), (1.19) it is simplest to determine the functions $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, the parameter $\mu'(\varepsilon)$ or $\mu(\varepsilon)$, and the coefficient $A_0(\varepsilon)$ in the form of series in powers of ε.

The results of the appropriate computations including terms containing ε^3 are as follows:

$$\Phi(\theta, \varepsilon) = \varepsilon C_{11} \sin\theta + \varepsilon^2 C_{22} \sin 2\theta + \varepsilon^3 (C_{13} \sin\theta + C_{33} \sin 3\theta)$$

$$\begin{aligned}\Psi^*(\theta, \varepsilon) = {} & \varepsilon \nu_1 C_{11} (\cos\theta - 1) + \varepsilon^2 [{}^1\!/_2 \nu_1^2 C_{11}^2 + \mu_0 C_{22} + \\ & + \mu_0 d_2)(\cos 2\theta - 1) - \nu_1^2 C_{11}^2 (\cos\theta - 1)] + \varepsilon^3 \Psi_3^*(\theta)\end{aligned} \qquad (2.13)$$

$$\begin{aligned}\mu(\varepsilon) = {} & \varepsilon \nu_1 \mu_0 C_{11} + \varepsilon^2 [(15\nu_1^2 + 27)\, {}^1\!/_{36} C_{11}^2 + {}^1\!/_2 \nu_2 C_{22}] + \\ & + \varepsilon^3 \mu_0 [{}^1\!/_{12} \nu_1 C_{11}^3 (\nu_1^2 + 9) + {}^1\!/_2 \nu_1 \nu_2 C_{11} C_{22} + \nu_1 C_{13} + {}^1\!/_3 \nu_3 C_{33}]\end{aligned}$$

$$A_{\sigma}(\varepsilon) = -\varepsilon^2 \, {}^1/_4 \, ({}^1/_9 \, \nu_1^2 - 1)$$

Here

$$C_{11} = \frac{d_1\mu_0}{\nu_1 - \mu_0}, \qquad C_{22} = \frac{1}{\nu_2 - \mu_0}({}^1/_2\nu_1^2 C_{11}^2 + \mu_0 d_2)$$
$$C_{13} = \frac{\mu_0}{\sqrt{\pi}}\frac{a_{13}}{\nu_1 - \mu_0}, \qquad C_{33} = \frac{\mu_0}{\sqrt{\pi}}\frac{a_{33}}{\nu_3 - \mu_0} \tag{2.14}$$

The polinomials a_{13} and a_{33} are linear with respect to the quantities C_{11}^3, C_{11}^2, C_{11}, C_{22} C_{11} and C_{22}; $\Psi_3^*(\theta)$ is a polynomial which is linear with respect to $\cos\theta - 1$, $\cos 2\theta - 1$, $\cos 3\theta - 1$ whose coefficients are linear polynomials in C_{13}, C_{33}, C_{11}^3, C_{11}^2, C_{22}, C_{11} C_{22}.

3. The solution in the case $\mu_0 = \nu_1$. In the general case of so-called branching it may turn out that $\mu_0 = \nu_n$. However, we shall limit ourselves to the example where $\mu_0 = \nu_1$. Direct computations show that in this case the solution must be constructed in the form of series in powers of $\varepsilon^{1/3}$. We can show this by an extension of the general methods of branching theory as developed by Liapunov and Schmidt.

Taking the first equation of the system in the form (2.1), we transform its kernel in accordance with Formula

$$K(\eta, \theta) = \frac{1}{\nu_1\pi}\sin\eta\sin\theta + \sum_{n=2}^{\infty}\frac{\sin n\eta \sin n\theta}{\nu_n} = \frac{1}{\nu_1\pi}\sin\eta\sin\theta + N(\eta, \theta) \tag{3.1}$$

The above equation then becomes

$$\Phi(\theta) = \mu_0(1+\mu')\int_0^{2\pi} N(\eta,\theta)[\Phi(\eta) + P(\eta)]\,d\eta +$$
$$+ \mu_0(1+\mu')\left[\xi\sin\theta + \frac{1}{\nu_1\pi}\int_0^{2\pi} P(\eta)\sin\eta\,d\eta\sin\theta\right] \tag{3.2}$$

Here

$$\xi = \frac{1}{\nu_1\pi}\int_0^{2\pi}\Phi(\eta)\sin\eta\,d\eta \tag{3.3}$$

According to the familiar lemma of Schmidt [5] the number $\mu_0 = \nu_1$ is not an eigenvalue of the kernel $N(\eta, \theta)$.

Equation (3.2) is equivalent to

$$\Phi(\theta) + P(\theta) = \mu_0(1+\mu')\int_0^{2\pi} N(\eta,\theta)[\Phi(\eta) + P(\eta)]\,d\eta + P(\theta) +$$
$$+ \mu_0(1+\mu')\left[\xi\sin\theta + \frac{1}{\nu_1\pi}\int_0^{2\pi} P(\eta)\sin\eta\,d\eta\sin\theta\right] \tag{3.4}$$

We denote the resolvent of the Fredholm second-order integral equation with the kernel $N(\eta, \theta)$ and the parameter $\mu_0(1+\mu')$ by $R_1[\eta, \theta, \mu_0(1+\mu')]$. Then, following the Liapunov-Schmidt method [5], we represent Eq. (3.4) in the equivalent form

$$\Phi(\theta) = \mu_0(1+\mu')\left[\xi\sin\theta + \frac{1}{\nu_1\pi}\int_0^{2\pi} P(\eta)\sin\eta\,d\eta\sin\theta\right] +$$
$$+ \mu_0(1+\mu')\int_0^{2\pi} R_1[\eta, \theta, \mu_0(1+\mu')] \times$$

$$\times \left\{P(\eta)+\mu_0(1+\mu')\left[\xi\sin\eta+\frac{1}{\nu_1\pi}\int_0^{2\pi}P(\eta_1)\sin\eta_1\,d\eta_1\sin\eta\right]\right\}d\eta \tag{3.5}$$

We note that Eq. (3.5) is valid for $\mu'=0$, since $\mu_0=\nu_1$ is not an eigenvalue of the kernel $N(\eta,\theta)$. As in the case of the resolvent R of Sect. 1, we write R_1 in explicit form,

$$R_1[\eta,\theta,\mu_0(1+\mu')]=\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{\sin n\eta\sin n\theta}{\nu_n-\mu_0(1+\mu')} \tag{3.6}$$

We leave the second equation of the system in its old form (2.1); we also leave Eqs. (1.19) and (1.26) in their old form.

As in Sect. 1, we first assume that μ' is a fixed quantity of small absolute value and solve the system consisting of Eqs. (3.5) and second Eq. (2.1) by the method of successive approximations. This yields the functions $\Phi(\theta,\varepsilon,\xi,\mu')$ and $\Psi^*(\theta,\varepsilon,\xi,\mu')$ as analytic functions of ε, μ' and ξ for $|\varepsilon|<\varepsilon_1''\leqslant\varepsilon_0$, $|\mu'|<\mu_1^\circ|$ and $|\xi|<\xi_1^\circ$. As in Sect. 1, we solve Eq. (1.26) for μ' and obtain $\mu'(\varepsilon,\xi)$ in the form of a series in powers of ε and ξ for $|\varepsilon|<\varepsilon_2''\leqslant\varepsilon_1''$ and $|\xi|<\xi_2^\circ\leqslant\xi_1^\circ$. This obviously implies the fulfillment of Eq. (3.5) and the second equation of (2.1) in which μ' was assumed to be a real quantity arbitrarily small in absolute value. From Eq. (1.19) we determine $A_0(\varepsilon,\xi)$ as a series in powers of ε and ξ for $|\varepsilon|<\varepsilon_3''\leqslant\varepsilon_2''$ and $|\xi|<\xi_3^\circ\leqslant\xi_2^\circ$.

According to the general theory of Liapunov and Schmidt [5], in order to convert from Eq. (3.5) to the initial first equation of (2.1) we must express the parameter ξ in terms of the parameter ε in such a way as to satisfy the branching equation obtainable from Eq. (3.3) by substituting into it the previously determined function $\Phi[\eta,\varepsilon,\xi,\mu'(\varepsilon,\xi)]$.

We recall that on substituting the function $\xi(\varepsilon)$ obtained from the branching equation into the expression for $\Phi[\eta,\varepsilon,\xi,\mu'(\varepsilon,\xi)]$, $\Psi^*[\eta,\varepsilon,\xi,\mu'(\varepsilon,\xi)]$, $\mu'(\varepsilon,\xi)$ and $A_0(\varepsilon,\xi)$, we have the solution of the initial system. Thus, according to the general theory the number and form of the solutions of the basic system as functions of ε are determined by the solutions of the branching equation. Let us construct the branching equation. As will become clear below, in our case we can stop with the terms containing ξ^3. To this end we must determine the following approximate expression from (3.2):

$$\Phi(\theta,\varepsilon,\xi)\approx\Phi_{01}(\theta)\xi+\Phi_{10}(\theta)\varepsilon+\Phi_{02}(\theta)\xi^2+\Phi_{11}(\theta)\xi\varepsilon+\Phi_{20}(\theta)\varepsilon^2+\Phi_{03}(\theta)\xi^3 \tag{3.7}$$

We must also find $\Psi^*(\theta,\varepsilon,\xi)$, $\mu(\varepsilon,\xi)=\mu_0(1+\mu')$, $A_0(\varepsilon,\xi)$. with the same degree of accuracy. Omitting all intervening computations and limiting ourselves to the term containing $\sin\theta$ required for the branching equation in the expression for Φ_{03}, we obtain

$$\Phi_{01}(\theta)=\mu_0\sin\theta,\quad \Phi_{10}(\theta)=\frac{\mu_0}{\nu_1}d_1\sin\theta,\quad \Phi_{02}(\theta)=\frac{\mu_0^4}{2(\nu_2-\mu_0)}\sin 2\theta$$

$$\Phi_{11}(\theta)=-2\mu_0 d_1\sin\theta+\frac{2\mu_0^3}{\nu_2-\mu_0}\sin 2\theta \tag{3.8}$$

$$\Phi_{20}(\theta)=-\frac{3\mu_0^2d_1^2}{\nu_1}\sin\theta+\frac{\mu_0(2\mu_0d_1^2+d_2)}{\nu_2-\mu_0}\sin 2\theta$$

$$\Phi_{03}(\theta)=\frac{\nu_1^3}{24(\nu_2-\nu_1)}\left[(15-8\nu_1^2)(\nu_2-\nu_1)+3\nu_1^2(2\nu_2-\nu_1)\right]\sin\theta$$

We have made allowance for the fact that $\mu_0=\nu_1$ in the expression for $\Phi_{03}(\theta)$. Taking account of values (3.8), we substitute the expression for Φ from (3.7) into (3.3). This yields the branching equation in the following approximate form:

$$d_1\varepsilon - 2d_1\nu_1\varepsilon\xi - 3\nu_1 d_1^2\varepsilon^2 + \alpha\xi^3 = 0 \tag{3.9}$$

$$\alpha = \frac{\nu_1^3}{24(\nu_2-\nu_1)}\left[(15-8\nu_1^2)(\nu_2-\nu_1)+3\nu_1^2(2\nu_2-\nu_1)\right] \tag{3.10}$$

Approximate equation (3.9) is sufficient for determining the number and form of all the small solutions of the complete branching equation. Constructing Newton's diagram for (3.9) [5], we see that its decreasing part consists of one segment (0, 1) and (3, 0). The branching equation therefore has three small solutions, each of which can be expressed as a convergent series in powers of $\varepsilon^{1/3}$. Since $\alpha > 0$, only one of these solutions is real. The problem under consideration therefore has a unique continuous small solution; this solution can be expressed as a convergent series in powers of $\varepsilon^{1/3}$.

We have therefore proved the following theorem.

Theorem 2. System (1.22), (1.21), (1.26), (1.19) for $\mu_0 = \nu_1$ has a unique solution $\Phi(\theta, \varepsilon)$, $\Psi(\theta, \varepsilon)$, $\mu(\varepsilon)$, $A_0(\varepsilon)$ which is small with respect to ε and continuous in θ $(0 \leqslant \theta \leqslant 2\pi)$. This solution can be expressed as series in powers of $\varepsilon^{1/3}$ which converge for $|\varepsilon| < \varepsilon_4'' \leqslant \varepsilon_3''$.

This theorem implies that in solving system (1.22), (1.21), (1.26), (1.19) for $\mu_0 = \nu_1$ it is simplest to determine directly the functions $\Phi(\theta, \varepsilon)$, $\Psi^*(\theta, \varepsilon) = \Psi(\theta, \varepsilon) - 1$, the parameter $\mu'(\varepsilon)$ or $\mu(\varepsilon)$, and the coefficient $A_0(\varepsilon)$ as series in powers of $\varepsilon^{1/3}$.

The results of the appropriate computations up to terms with $\varepsilon^{2/3}$ are as follows:

$$\Phi(\theta, \varepsilon) = -\varepsilon^{1/3}(d_1\beta)^{1/3}\sin\theta + \varepsilon^{2/3}(d_1\beta)^{2/3}\frac{\nu_1}{2(\nu_2-\nu_1)}\sin 2\theta$$

$$\Psi^*(\theta, \varepsilon) = -\varepsilon^{1/3}(d_1\beta)^{1/3}(\cos\theta - 1) + \varepsilon^{2/3}(d_1\beta)^{2/3}\nu_1^2\left[1 - \cos\theta + \frac{2\nu_2-\nu_1}{4(\nu_2-\nu_1)}(\cos 2\theta - 1)\right] \tag{3.11}$$

$$\mu(\varepsilon) = -\varepsilon^{1/3}(d_1\beta)^{1/3}\nu_1^2 + \varepsilon^{2/3}(d_1\beta)^{2/3}\frac{\nu_1}{12(\nu_2-\nu_1)}\left[(5\nu_1^2+9)(\nu_2-\nu_1)+3\nu_2\nu_1^2\right]$$

$$A_0(\varepsilon) = -\varepsilon^{2/3}(d_1\beta)^{2/3}\,{}^1\!/_4\left(\frac{\nu_1^2}{9} - 1\right) \qquad \left(\beta = \frac{\nu_1^3}{\alpha}\right)$$

4. Determination of the wave profile. The wave profile is given parametrically by Eqs. (1.14). Substituting the resulting functions $\tau(\theta, \varepsilon)$ and $\Phi(\theta, \varepsilon)$ into these equations and eliminating θ, we obtain the equation of the profile in the form $y = y(x, \varepsilon)$.

The profile equations to within second-order terms for the two cases considered above are as follows:

for $\mu_0 \neq \nu_n$,

$$y(x, \varepsilon) = \frac{1}{k}\left[\varepsilon C_{11}(\cos kx - 1) + \frac{1}{2}\varepsilon^2\left(\frac{1}{6}\nu_1 C_{11}^2 - C_{22}\right)(1 - \cos 2kx)\right]\left(k = \frac{2\pi}{\lambda}\right)$$

where C_{11} and C_{22} are given by Formulas (2.14);

for $\mu_0 = \nu_1$,

$$y(x, \varepsilon) = \frac{1}{k}\left[-\varepsilon^{1/3}(d_1\beta)^{1/3}(\cos kx - 1) + {}^1\!/_2\,\varepsilon^{2/3}(d_1\beta)^{2/3}\frac{\nu_1^2(\nu_2 - 4\nu_1)}{6(\nu_2-\nu_1)}(1 - \cos 2kx)\right]$$

As stipulated, the origin lies at the wave crest. Hence, on setting $\nu_1 < \mu_0 < \nu_2$ and analyzing the principal terms in the formulas for $y = y(x, \varepsilon)$, we conclude that we must set $d_1 < 0$.

Finally, we note that $\mu_0 = \nu_1$ is the special case mentioned at the beginning of the paper.

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Translated by A. Y.

SMALL TORSIONAL OSCILLATIONS OF AN ELASTICALLY CONSTRAINED RIGID CIRCULAR CYLINDER FILLED WITH A VISCOUS FLUID

PMM Vol. 32, №6, 1968, pp. 1026-1033

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(Received July 18, 1968)

This paper considers the rotatory motion of a cylinder of finite height, filled with a viscous incompressible fluid, and subjected to an elastic moment, the cylinder being initially at rest in a position obtained from the equilibrium position by rotation through a small angle . The solution of the problem is constructed in the form of a Laplace-Mellin integral. The vibration spectrum of the system is studied and a spectral expansion for the solution obtained, the latter yielding a description of the nature of the cylinder's motion for various values of the parameters involved.

This problem was solved earlier under the assumption that the oscillations decay harmonically, which assumption is valid in a certain time interval when the ratio of the moments caused by the viscous friction forces to the maximal elastic moment is sufficiently small [1 and 2]. A general investigation of the characteristic equation for the oscillations was not carried out, and the problem in the large (with account taken of the initial conditions) was not posed. The present paper fills this gap.

It is established that for any positive values of the parameters the rigid cylinder passes through the equilibrium position. Depending on the values of the parameters, two things can happen: (1) the cylinder passes through the equilibrium position an infinite number of times, or (2) the cylinder passes through the equilibrium position an odd number of times and then approaches the equilibrium position as time approaches infinity, from the side opposite that of the initial position.